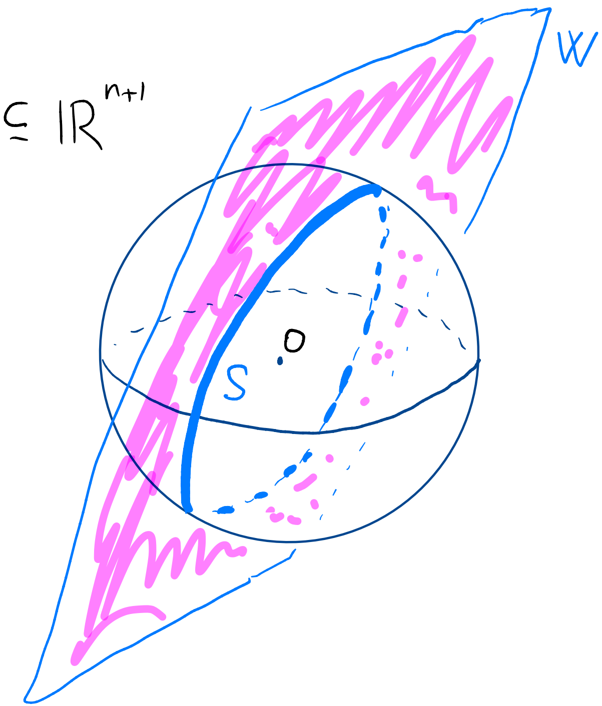
Higher-dimensional hyperbolic manifolds via Coxeter polytopes

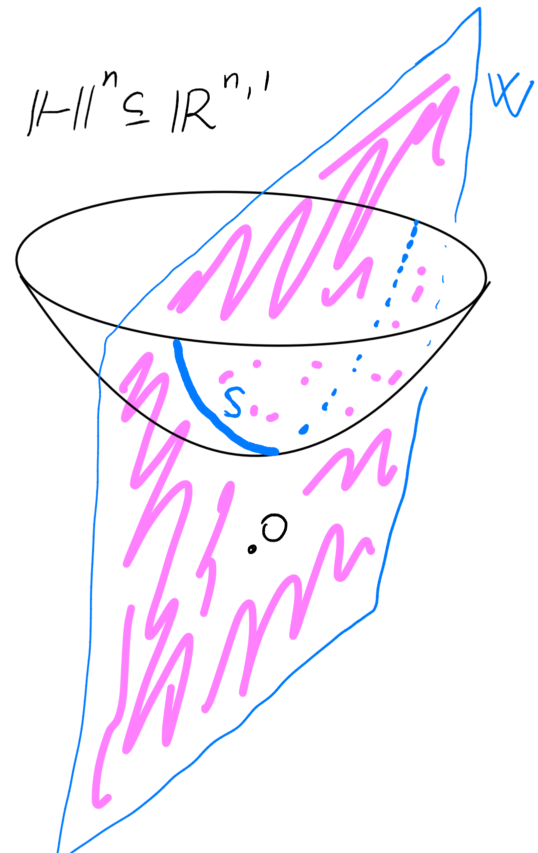
Ventotene 2025

$\mathbb{X}^n = \mathbb{H}^n, \mathbb{R}^n, \mathbb{S}^n$ K-dimensional SUBSPACES

$\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$



$\mathbb{H}^n \subseteq \mathbb{R}^{n,1}$

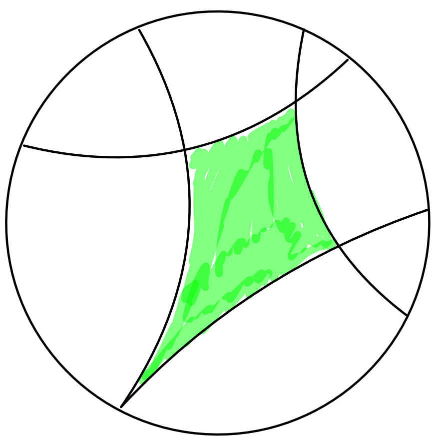


$S = \mathbb{X}^n \cap W$

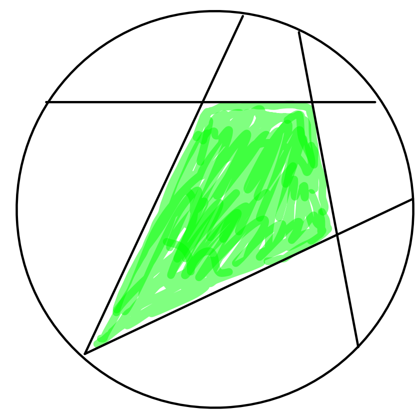
A POLYHEDRON is the intersection $P = H^1 \cap \dots \cap H^k \subseteq \mathbb{X}^n$ of finitely many half-spaces $H^1, \dots, H^k$

such that

1) $\mathrm{int}\, P \neq \emptyset$
2) $\mathrm{vol}(P) < +\infty$

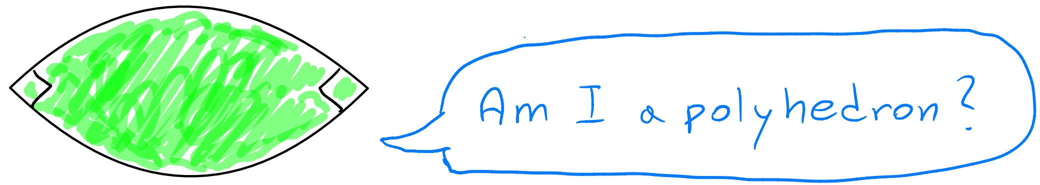


Poincaré model



Klein model

Some care is needed in $\mathbb{S}^n$ if we allow P to contain antipodal points



A FACE in P is $F = P \cap \partial H$ with $H \supseteq P$ halfspace

It is a polyhedron in its SUPPORT SUBSPACE

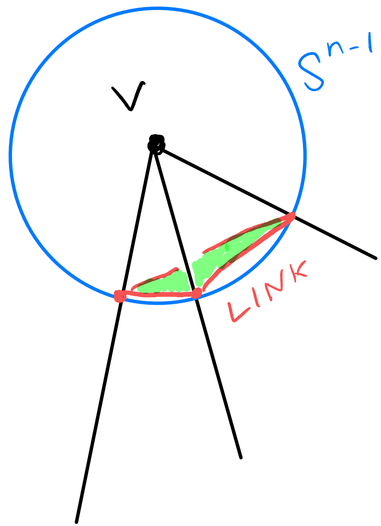
smallest $S \subseteq \mathbb{X}^n$ that contains F

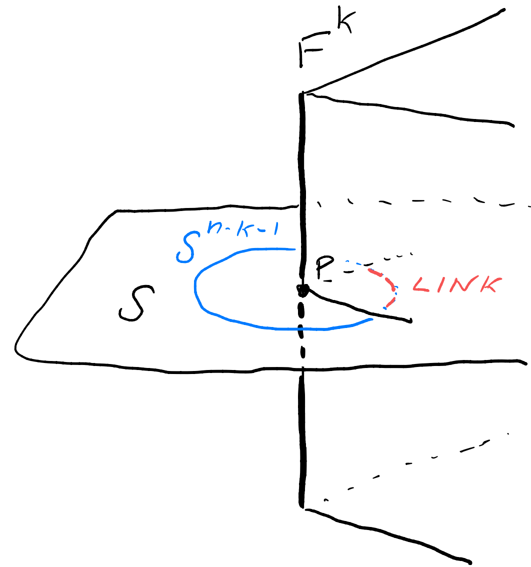
It has dimension $K = 0, 1, \ldots, n-2, n-1$

vertex, edge, ridge, facet

Ex: P is the convex hull of its vertices in $\overline{\mathbb{H}}^n, \mathbb{R}^n, (\mathbb{S}^n)$

The LINK of a face F of P:





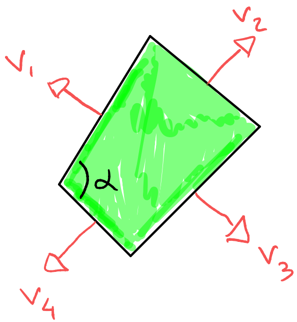
is a spherical $(n-k-1)$-polyhedron

If F is a ridge, the link is an arc of length $\alpha < \pi$

DIHEDRAL ANGLE of F

Gram matrix

$\mathbb{R}^n$



$$G_{ij} = \langle v_i , v_j \rangle$$

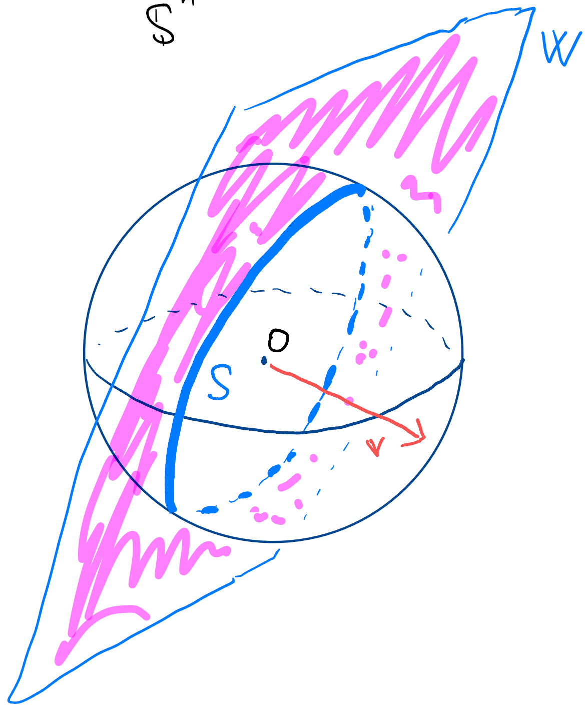
$$G_{ii} = 1$$

$i \neq j$:

$$G_{ij} = \begin{cases} -1 & \text{if parallel} \\ -\cos\alpha & \text{if incident} \end{cases}$$

$\mathrm{rk}\, G = n$, $\sigma = (\underset{+}{n}, \underset{-}{0}, \underset{0}{k})$

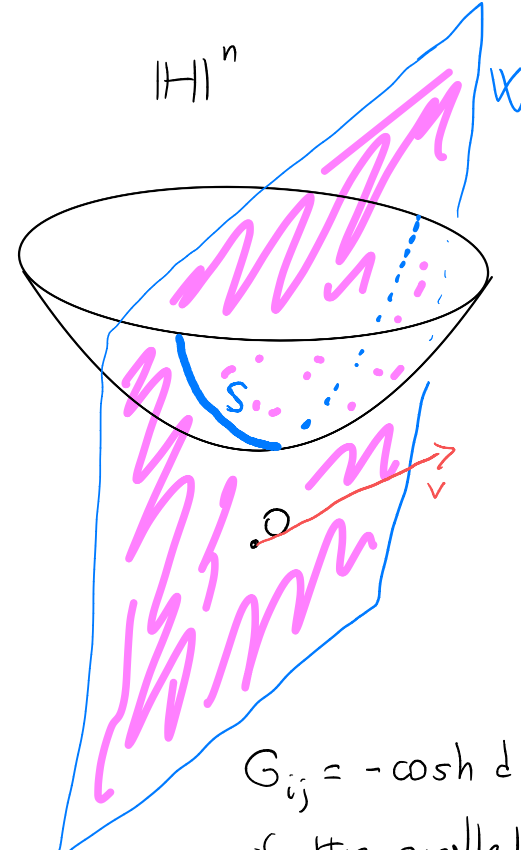
$\mathbb{S}^n$



$\mathrm{rk}\, G = n+1$

$\sigma = (n+1, 0, k)$

$\mathbb{H}^n$



$G_{ij} = -\cosh d$

if ultraparallel

$\mathrm{rk}\, G = n+1$

$\sigma = (n, 1, k)$

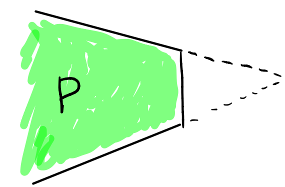
P is NON OBTUSE if dihedral angles are all $\leq \frac{\pi}{2}$

$$\iff G_{ij} \leq 0 \quad \forall i \neq j$$

[Andreev] If P non obtuse,

$$\partial H_{i_1} \cap \dots \cap \partial H_{i_k} \neq \emptyset \iff \text{corresponding facets intersect}$$

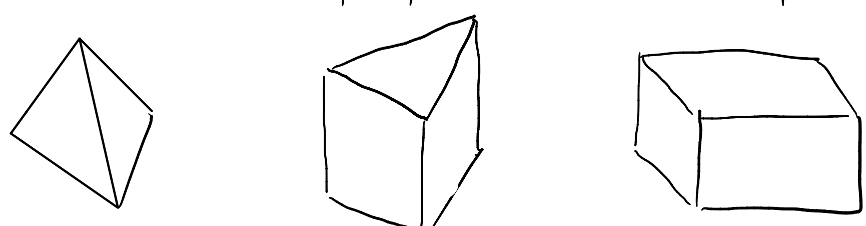
NO



[Vinberg?]

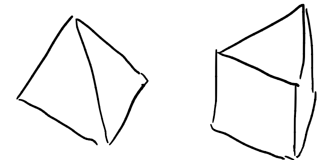
Every non obtuse spherical polyhedron is a simplex

Every non obtuse Euclidean polyhedron is a product of simplexes



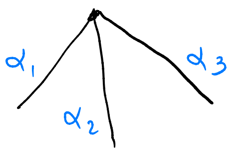
Cor: Every non-obtuse P is SIMPLE, except at the ideal points, whose link is a product of simplexes.

all links are simplexes

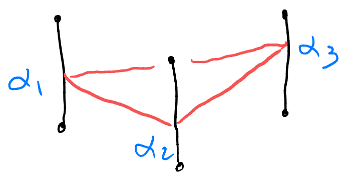
[Andreev] P simple 3-dimensional, $P \neq$ 

$0 < \alpha_i \leq \frac{\pi}{2}$ dihedral angles assigned to ridges

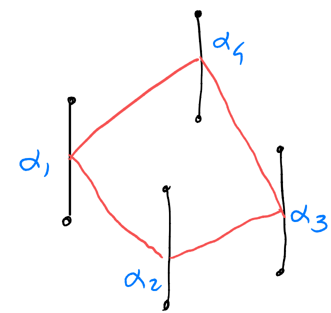
$\exists$ realization $P \subseteq \mathbb{H}^3$ with these dihedral angles $\iff$

1) 

$\alpha_1 + \alpha_2 + \alpha_3 > \pi$

2) 

$\alpha_1 + \alpha_2 + \alpha_3 < \pi$

3) 

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq (\pi/2, \pi/2, \pi/2, \pi/2)$

[Roeder, Hubbard, Dunbar]

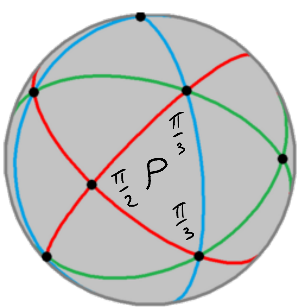
Coxeter polyhedra

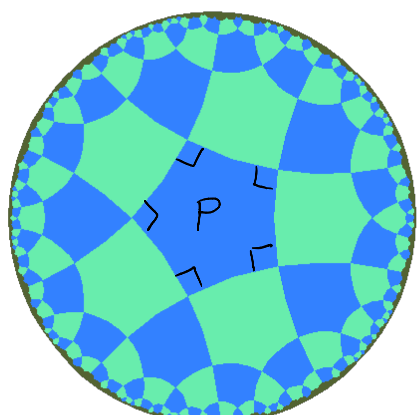
A COXETER POLYHEDRON is a polyhedron P whose dihedral angles are $\alpha_{ij} = \frac{\pi}{n_{ij}}$, $n_i \geq 2$ $\quad \forall\, F_i \cap F_j \neq \emptyset$

Thm: $R_i :=$ reflection along facet F_i $\quad R_i \in \mathrm{Isom}(\mathbb{X}^n)$

$\Gamma := \langle R_1, \dots, R_k \rangle < \mathrm{Isom}(\mathbb{X}^n)$ discrete with fundamental domain P

$\Gamma = \langle R_1, \dots, R_k \mid R_i^2, (R_i R_j)^{n_{ij}} \rangle$





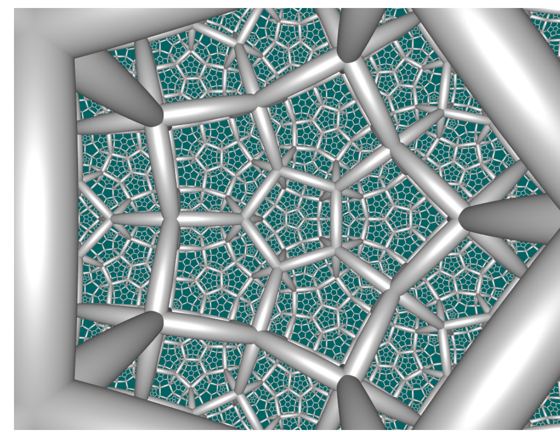


Image by Roice3

$P_1 \subseteq \mathbb{R}^{n_1} \quad P_2 \subseteq \mathbb{R}^{n_2} \quad \rightsquigarrow \quad P_1 \times P_2 \subseteq \mathbb{R}^{n_1+n_2}$ PRODUCT

$P_1 \subseteq S^{n_1} \quad P_2 \subseteq S^{n_2} \quad \rightsquigarrow \quad P_1 * P_2 \subseteq S^{n_1+n_2+1}$ JOIN

$$P_1 * P_2 = \mathrm{lk}\,(C(P_1) \times C(P_2))$$

$P \subseteq \mathbb{R}^n, S^n$ is IRREDUCIBLE if it is not a product or join of two smaller dimensional polyhedra

$\Downarrow$

SIMPLEX

[Coxeter] The irreducible Coxeter spherical simplexes are

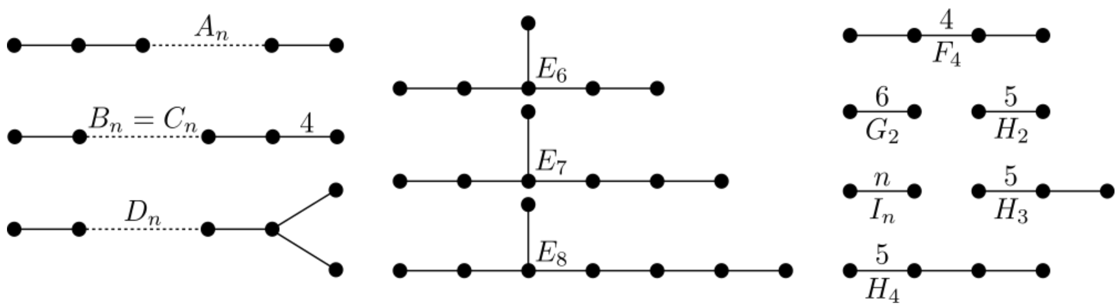


Image by Rgugliel

The flat Coxeter simplexes are

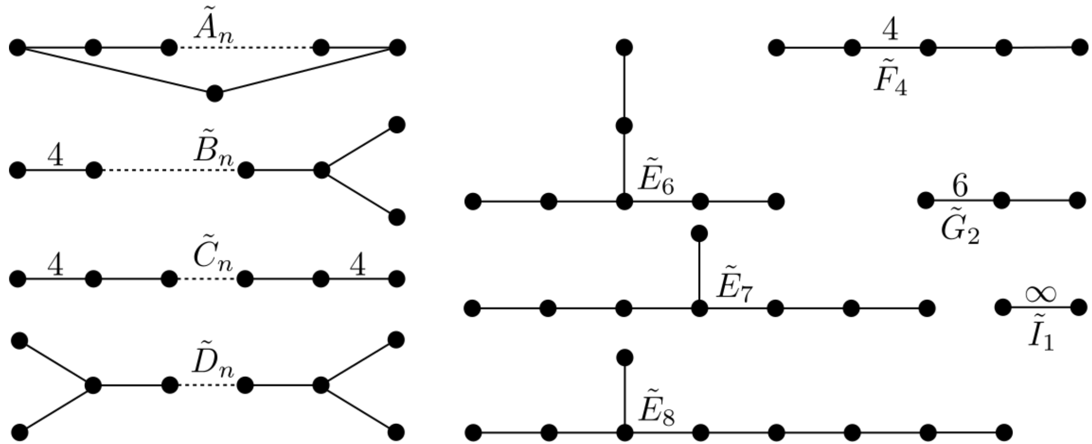
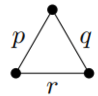


Image by Rguglie1

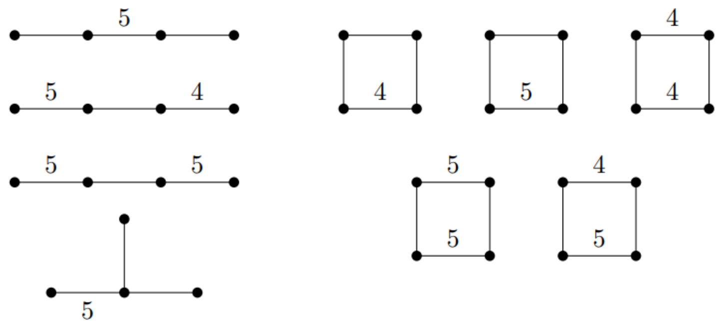
[Lanner] The compact hyperbolic simplexes are

$n = 2$



with $\beta = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$

$n = 3$



$n = 4$

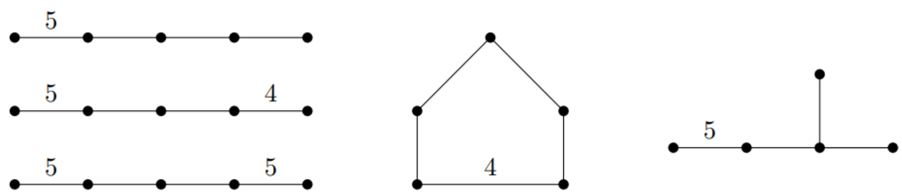
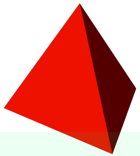
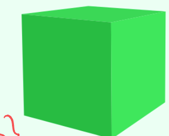
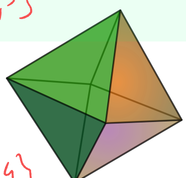
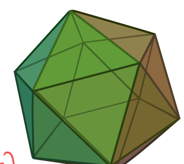
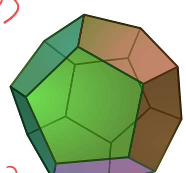


Table E.2: The Coxeter diagrams corresponding to compact hyperbolic Coxeter systems.

Regular polyhedra	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
 $\{3,3\}$	$\mathbb{H}^3$ ideal	$\mathbb{S}^3$	$\mathbb{S}^3$	$\mathbb{S}^3$
 $\{4,3\}$	$\mathbb{H}^3$ ideal	$\mathbb{H}^3$ $\{4,3,5\}$	$\mathbb{R}^3$	$\mathbb{S}^3$
 $\{3,4\}$			$\mathbb{H}^3$ ideal	$\mathbb{S}^3$
 $\{3,5\}$				$\mathbb{H}^3$
 $\{5,3\}$	$\mathbb{H}^3$ ideal	$\mathbb{H}^3$	$\mathbb{H}^3$ $\{5,3,4\}$	$\mathbb{S}^3$

Schläfli notation

$\{n\}$ regular n-gon

$\{p,q\}$ regular p-gons meeting q times at vertices

$\{p,q,r\}$ $\{p,q\}$'s meeting r times at edges

$\{5,3,4\}$ $\circ\overset{5}{\text{———}}\circ\overset{3}{\text{———}}\circ\overset{4}{\text{———}}\circ$

$\{4,3,5\}$ $\circ\overset{4}{\text{———}}\circ\overset{3}{\text{———}}\circ\overset{5}{\text{———}}\circ$

Regular polytopes	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
4-simplex $\{3,3,3\}$		$\mathbb{H}^4$	$\mathbb{S}^4$	$\mathbb{S}^4$
hypercube $\{4,3,3\}$		$\mathbb{H}^4$ $\{4,3,3,5\}$	$\mathbb{R}^4$	$\mathbb{S}^4$
16-cell $\{3,3,4\}$				$\mathbb{R}^4$
24-cell $\{3,4,3\}$			$\mathbb{H}^4$ ideal	$\mathbb{R}^4$
120-cell $\{5,3,3\}$		$\mathbb{H}^4$	$\mathbb{H}^4$ $\{5,3,3,4\}$	$\mathbb{H}^4$
600-cell $\{3,3,5\}$				
32-cell				$\mathbb{H}^5$ ideal

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subseteq S^3$$

$$\pi : S^3 \to SO(3)$$

Binary group $G_{2n}^* = \pi^{-1}(G_n)$

BINARY TETRAHEDRAL GROUP:

$$T_{24}^* = Q_8 \cup \left\{\pm\frac{1}{2} \pm \frac{i}{2} \pm \frac{j}{2} \pm \frac{k}{2}\right\}$$

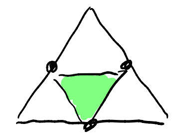
BINARY ICOSAHEDRAL GROUP.

$$I_{120}^* = T_{24}^* \cup \{\dots\}$$

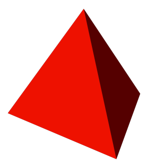
$$Q_8 < T_{24}^* < I_{120}^*$$

A polyhedron P is **SEMIREGULAR** if $\mathrm{Isom}(P)$ acts transitively on vertices, and all facets are regular.

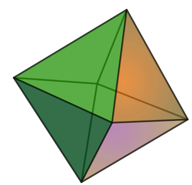
Ideal hyperbolic RECTIFIED 4-SIMPLEX



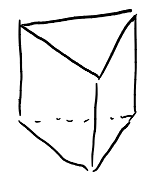
facets: 5



5

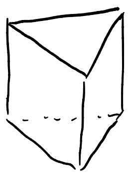


link(v):



Dihedral angles $\pi/2$ and $\pi/3$

Gosset polytopes (1900)



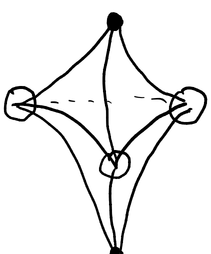
G^3	G^4	G^5	G^6	G^7	G^8
	rectified 4-simplex	5-demicube	27 vertices	56 vertices	240 vertices

$=$ non-zero elements in $E_8 \subseteq \mathbb{R}^8$ of smallest norm

FACETS ARE SIMPLEXES AND CROSS-POLYTOPES

Dual right-angled hyperbolic polytopes

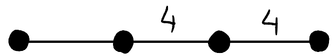
[Agol, Long, Reid] [Potyagailo-Vinberg] [Ratcliffe-Tschantz]



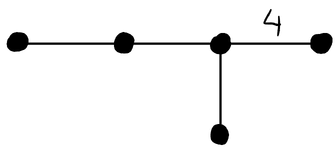
P^3	P^4	P^5	P^6	P^7	P^8
	10 facets	16	27	56	240
	5 ideal vertices	10	27	126	2160
	5 real vertices	16	72	576	17280

How they were discovered:

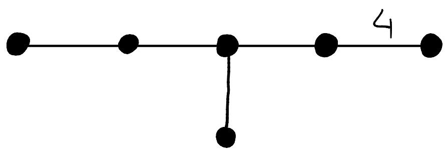
n = 3



n = 4

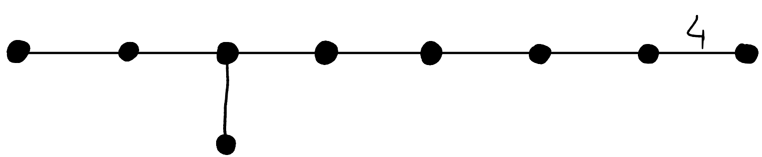


n = 5

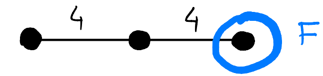


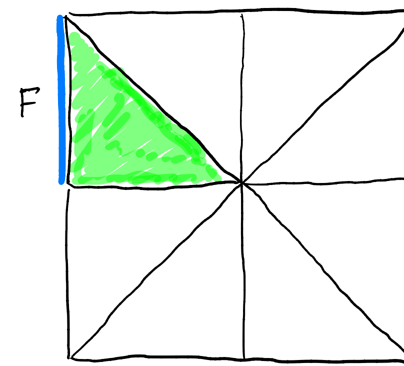
⋮

n = 8



n = 2





Hyperbolic simplexes with one ideal vertex